Strong Induction

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These slides are mainly taken from http://www.cs.laurentian.ca/jdompierre/html/MATH2056E_W2011/index.html

Definition: Strong Induction

If $P(n)$, be a predicate on a well-ordered set S. Then the rule of inference

$$\begin{array}{rl} & P(0) \\ & \forall k[(P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k+1)] \\ \hline \therefore & \forall n\, P(n) \end{array}$$

is known as the **complete induction** (or the **strong induction**, or the **second principle of mathematical induction**).

Example of Strong Induction

PRELIMINARY STEP: Let $S = \mathbb{N} - \{0, 1\}$ and let $P(n)$ be the propositional function: "*n can be written as the product of prime numbers*".

BASIS STEP: We verify that the proposition $P(2)$ is true.

INDUCTIVE STEP: Suppose that $P(j)$ is true for all positive integers j where $2 \leq j \leq k$. According to this hypothesis, we must show that $P(k+1)$ is true.

Therefore, we can conclude that $P(n)$ is true for all integers n in the set S.

Remarks

The proposition $\forall k[(P(1) \wedge P(2) \wedge \ldots \wedge P(k)) \to P(k+1)]$ is generally proved by universal instantiation. We prove that the proposition $(P(1) \wedge P(2) \wedge \ldots \wedge P(k)) \to P(k+1)$ is true for an arbitrary k, without any other hypothesis on k. Then, by universal generalization, the proposition is true $\forall k$.

The mathematical induction and strong induction are equivalent. Any proof using mathematical induction can also be considered to be a proof by strong induction because the inductive hypothesis of a proof by mathematical induction is a part of the inductive hypothesis of a proof by strong induction. However, it is much more awkward to convert a proof by strong induction into a proof by mathematical induction, but it's possible.

The Well-Ordering Property

The **well-ordering property**: Every non empty subset of the set of positive integers has a least element.

The mathematical induction, the strong induction and the well-ordering property are equivalent. The well-ordering property can be used directly in proofs.